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Towards the classification of (P and Q)-polynomial association schemes

Tatsuro Ito

Anhui University

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at USTC

Hamming scheme $H(n,q)$

$$X = F^n = \underbrace{F \times \cdots \times F}_{n}, \qquad |F| = q$$

$$x = (x_1, \cdots, x_n), \qquad x_i \in F$$
$$y = (y_1, \cdots, y_n), \qquad y_i \in F$$

$$\partial(x,y) = \#\{\, i \mid x_i \neq y_i, \quad 1 \leq i \leq n \,\}$$

metric on X

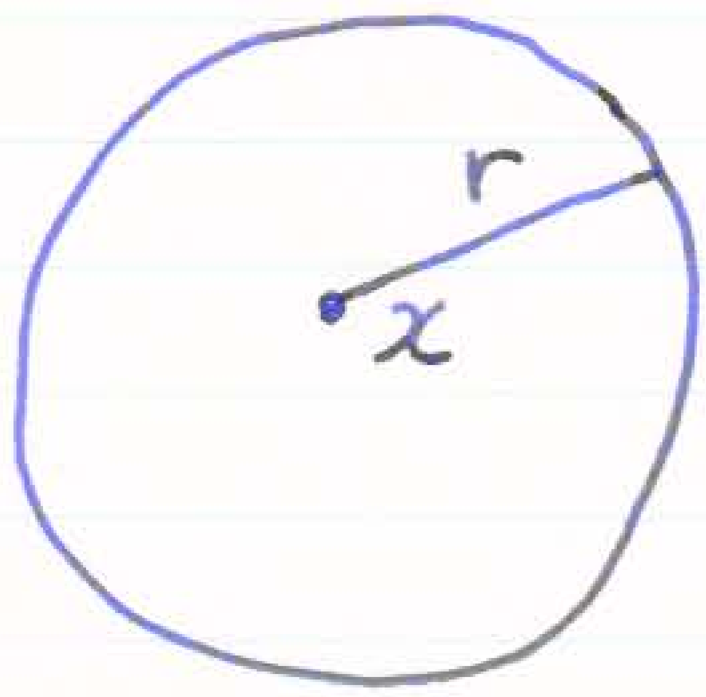
$$A_i : X \times X \longrightarrow \{0,1\}$$
$$(x,y) \longmapsto A_i(x,y) = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{else} \end{cases}$$

$\exists\ v_i(x)$ polynomial of degree i
(Krawchouk polynomial)

$$A_i = v_i(A_1) \quad \text{in} \quad M_X(\mathbb{C})$$

the full matrix alg.

$$B_r(x) = \{ y \in X \mid \partial(x,y) \leq r \}$$



r- ball
with centre $x \in X$

$$|B_r(x)| = 1 + k + v_2(k) + \cdots + v_r(k)$$

where $k = (q-1)n = |B_1(x)| - 1$

$X \supset Y$ <u>e- code</u>

if $B_e(x) \cap B_e(y) = \phi$ for $x, y \in Y$, $x \neq y$

equivalently

$${}^t\phi A_i \phi = 0 \quad \text{for } 1 \leq i \leq 2e$$

where $\phi : X \longrightarrow \{0,1\}$
$x \longmapsto \phi(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{else} \end{cases}$

characteristic vector of Y

sphere packing bound

$$|Y| \leq \frac{|X|}{1 + k + v_2(k) + \cdots + v_e(k)}$$

for an e-code Y.

Y : perfect e-code
if the equality holds.

Lloyd Theorem

Set $\Phi_e(z) = 1 + z + v_2(z) + \cdots + v_e(z)$.

If Y is a perfect e-code,
i.e. $|Y| = \Phi_n(k) / \Phi_e(k)$, $k = (q-1)n$,

then
$$\Phi_e(z) \mid \Phi_n(z) \quad \text{in } \mathbb{C}[z].$$

$\Phi_e(z)$: Lloyd polynomial

DTG <u>distance-transitive graph</u>

D.G. Higman, N.L. Biggs
in 60s

$\Gamma = (X, R)$ finite simple graph
pts edges connected

$\partial(x, y)$: distance between $x, y \in X$
the length of a shortest path
joining x, y

Γ is <u>distance-transitive</u> (DT)

iff $\forall x, y, x', y' \in X$
$\partial(x, y) = \partial(x', y')$

$\exists \sigma \in Aut(\Gamma)$

$x^\sigma = x', \quad y^\sigma = y'$

No. code
Date DRG

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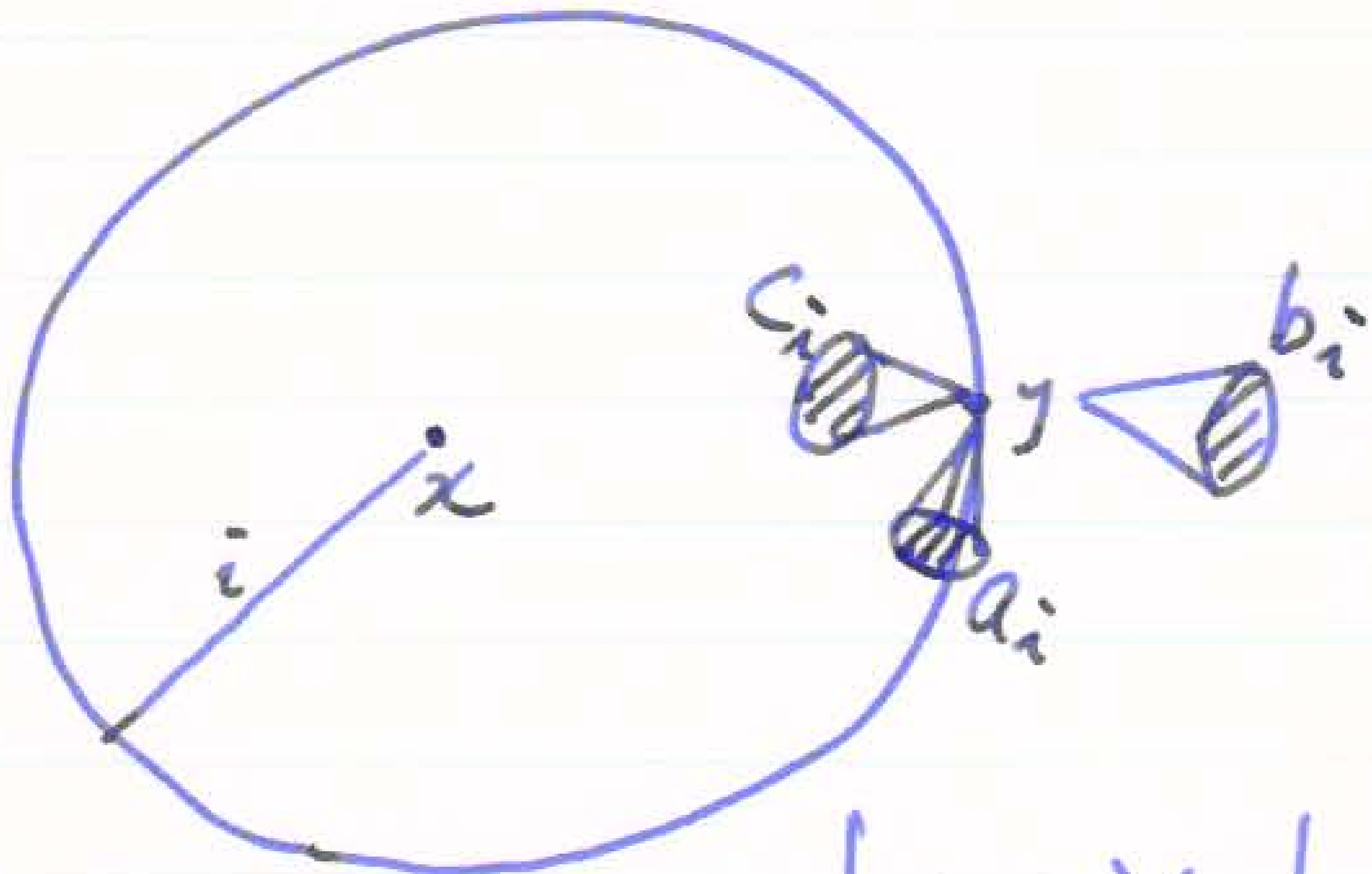
DTG $\Longrightarrow$ DRG distance - regular graph



$$R_i(x) = \{ y \in X \mid \partial(x, y) = i \}$$

$\forall x \in X$

$\forall y \in R_i(x)$

$$c_i = | R_{i-1}(x) \cap R_1(y) |$$

$$a_i = | R_i(x) \cap R_1(y) |$$

$$b_i = | R_{i+1}(x) \cap R_1(y) |$$

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$\Gamma = (X, R)$ DRG

$$d = \max\{\partial(x,y) \mid x, y \in X,\ x \neq y\}$$

diameter

$$A_i : X \times X \longrightarrow \{0, 1\}$$

$$(x,y) \longmapsto A_i(x,y) = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{else} \end{cases}$$

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}$$

Define polynomials $v_i(x)$ of degree i, $0 \leq i \leq d$ by $v_0(x) = 1$, $v_1(x) = x$,

$$x\, v_i(x) = b_{i-1} v_{i-1}(x) + a_i v_i(x) + c_{i+1} v_{i+1}(x).$$

Then

$$A_i = v_i(A_1), \quad 0 \leq i \leq d.$$

$\{v_i(x)\}_{i=0}^{d}$: orthogonal polynomials

N. Biggs : Lloyd type theorem of codes in a DRG

$\Gamma = (X, R)$ DRG

$X \supset Y$: <u>e-code</u>

if $B_e(x) \cap B_e(y) = \phi$ for $x, y \in Y$, $x \neq y$

$|B_e(x)| = 1 + k + v_2(k) + \cdots + v_e(k)$,

$k = |B_1(x)| = b_0$, valency of Γ

<u>sphere packing bound</u>

$$|Y| \leq \frac{|X|}{1 + k + v_2(k) + \cdots + v_e(k)}$$

for an e-code Y.

<u>Lloyd type theorem</u>

Set $\Phi_e(z) = 1 + z + v_2(z) + \cdots + v_e(z)$.

If Y is a perfect e-code,

i.e. $|Y| = \Phi_d(k) / \Phi_e(k)$,

then $\Phi_e(z) \mid \Phi_d(z)$ in $\mathbb{C}[z]$.

Johnson scheme $J(v, k)$, $k \leq \frac{v}{2}$

$X = \binom{V}{k}$, $|V| = v$

the set of k-subsets of V

$X \ni x, y$

$\partial(x, y) = k - |x \cap y|$ metric on X

DT distance-transitive

Incidence structure between $J(v,k)$
and $J(v,t)$

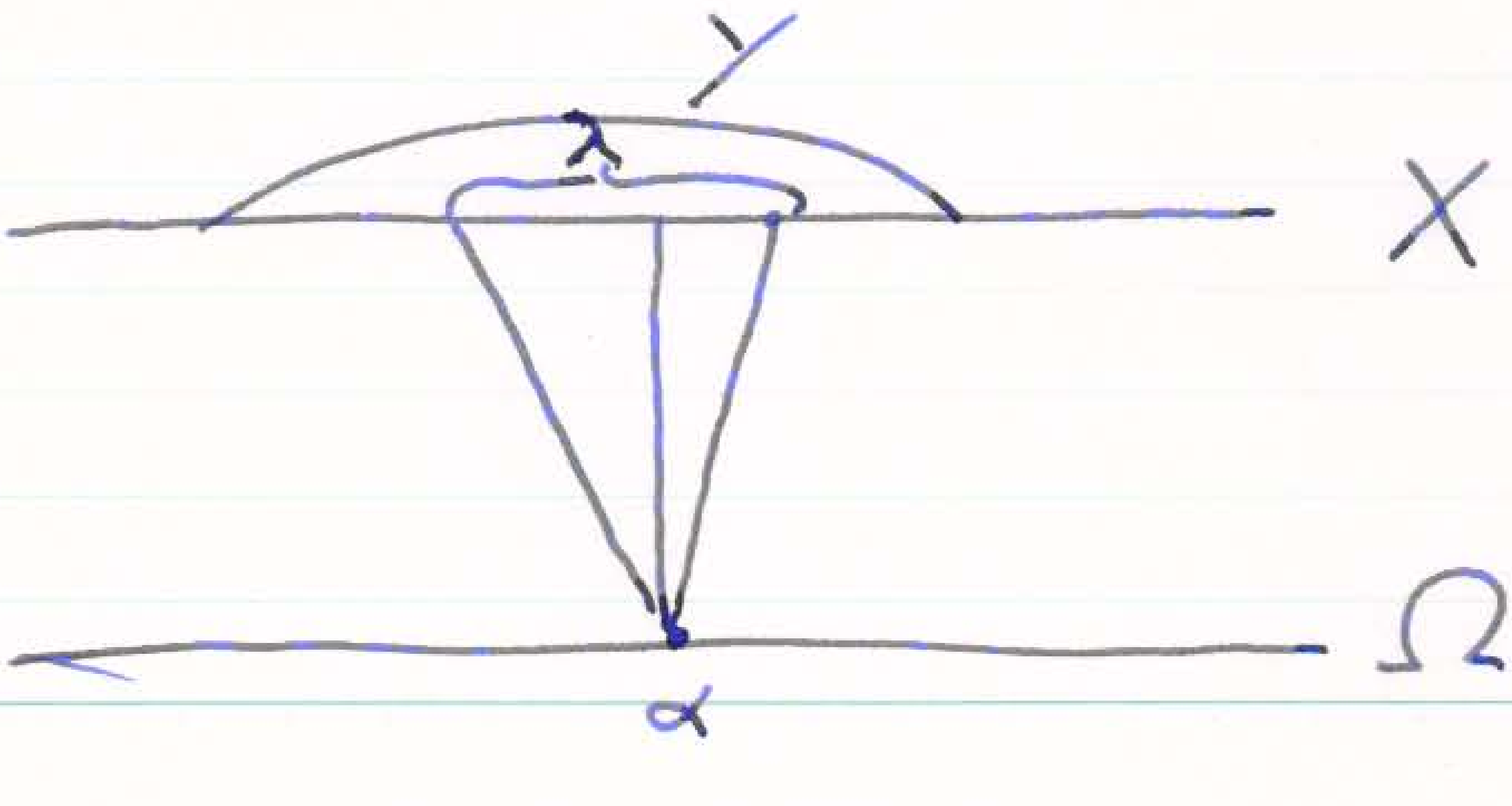
$X = \binom{V}{k}$.

$\Omega = \binom{V}{t}$. $\quad t \leq k$

$X \ni x$ covers $\alpha \in \Omega$
if $x \supseteq \alpha$ as subsets of V.

$X \supseteq Y$: $t-(v,k,\lambda)$ design

if $\#\{y \in Y \mid y \text{ covers } \alpha\} = \lambda_\alpha = \lambda$
for every $\alpha \in \Omega$.



Fisher's inequality

$t = 2$. $\qquad Y : 2\text{-}(v, k, \lambda)$ design

$$|Y| \geq v$$

RayChaudhuri - Wilson

around 1970

$Y : t\text{-}(v, k, \lambda)$ design

$$|Y| \geq \binom{v}{e}, \qquad e = \left[\frac{t}{2}\right]$$

Y is tight if the equality holds.

In this case, $\quad t = 2e$.

$\{v_i^*(z)\}_{i=0}^{k}$, $\deg v_i^*(z) = i$

Hahn polynomials

$$\Phi_e^*(z) = \sum_{i=0}^{e} v_i^*(z)$$

Wilson polynomial

Fisher type inequality

$$|Y| \geq \Phi_e^*(m), \qquad m = v-1$$

$$\left(\Phi_e^*(m) = \binom{v}{e} \right)$$

Lloyd type theorem

If the equality holds above,

then

$$\Phi_e^*(z) \;\Big|\; \Phi_k^*(z) \qquad \text{in } \mathbb{C}[z]$$

P. Delsarte, An algebraic approach to the association schemes of the coding theory, Thesis, Université Catholique de Louvain (1973), Philips Res. Repts Suppl. 10 (1973)

$\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ symmetric association scheme

$X \times X \supset R_i, \quad 0 \le i \le d$

$$A_i : X \times X \longrightarrow \{0, 1\}$$
$$(x, y) \longmapsto A_i(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{else} \end{cases}$$

$$\begin{cases} A_0 = I & \text{identity} \\ A_0 + A_1 + \cdots + A_d = J & \text{all one matrix} \\ {}^tA_i = A_i, \ 0 \le i \le d & \text{symmetric} \\ A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k & \end{cases}$$

$\mathfrak{A} = \langle A_0, A_1, \cdots, A_d \rangle \subset M_X(\mathbb{C})$

the full matrix algebra

Bose-Mesner algebra

$\dim \mathfrak{A} = d+1$

Def $\mathfrak{X}$ is P-polynomial

if $\exists\, v_i(x)$ polynomial of degree i

$$A_i = v_i(A_1), \quad 0 \le i \le d$$

Fact

$\mathfrak{X}$: P-polynomial

$\Longleftrightarrow$

$\Gamma = (X, R_1)$: DRG

In this case,

$R_i \ni (x,y) \Longleftrightarrow \partial(x,y) = i$ in Γ

<u>duality</u> for a symmetric association scheme

$\mathfrak{A} = \langle A_0, A_1, \cdots, A_d \rangle$ BM-alg

commutative
semi-simple

$= \langle E_0, E_1, \cdots, E_d \rangle$, $\quad E_0 = \frac{1}{|X|} J$

primitive idempotents

$$E_0 + E_1 + \cdots + E_d = I$$

$$E_i E_j = \delta_{ij} E_i$$

$\mathfrak{A}$ is closed under the Hadamard product $\circ$

(entry-wise product)

$A_0 + A_1 + \cdots + A_d = J$ all one matrix
the identity w.r.t. $\circ$

$$A_i \circ A_j = \delta_{ij} A_i$$

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k E_k$$

(q_{ij}^k : the Krein parameter)

<u>Def</u> $\mathcal{X}$ is Q-polynomial

if $\exists\, v_i^*(x)$ polynomial of degree i

$$nE_i = v_i^*(nE_1), \quad 0 \le i \le d \qquad (n = |X|)$$

w.r.t. the Hadamard product $\circ$

<u>Fact</u> $\mathcal{X}$: Q-polynomial

$\Longleftrightarrow$

$$(nE_1) \circ (nE_i) = b_{i-1}^* (nE_{i-1}) + a_i^* (nE_i) + c_{i+1}^* (nE_{i+1})$$

$$, 0 \le i \le d$$

$\{v_i^*(x)\}_{i=0}^{d}$: orthogonal polynomials

$$x\, v_i^*(x) = b_{i-1}^*\, v_{i-1}^*(x) + a_i^*\, v_i^*(x) + c_{i+1}^*\, v_{i+1}^*(x)$$

$$\left(v_0^*(x) = 1, \quad v_1^*(x) = x \right)$$

Observation

$\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ P-poly.

$X \supset Y$: e-code

$\Longleftrightarrow$

$${}^t\phi A_i \phi = 0, \quad 1 \le i \le 2e$$

where ϕ is the characteristic vector of Y

Definition

$\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ Q-poly.

$X \supset Y$: t-design

$\Longleftrightarrow$

$${}^t\phi E_i \phi = 0, \quad 1 \le i \le t$$

where ϕ is the characteristic vector of Y.

Fisher type inequality

$X \supset Y$: t-design

Then

$$|Y| \geq v_0^*(m) + v_1^*(m) + \cdots + v_e^*(m),$$

$$\left(e = \left[\tfrac{t}{2}\right], \qquad m = \operatorname{rank} E_1 \right)$$

Y is a <u>tight t-design</u>

if the equality holds.

Set $\Phi_e^*(z) = v_0^*(z) + v_1^*(z) + \cdots + v_e^*(z)$

Wilson polynomial

<u>Theorem</u>

Y : tight t-design

Then $t = 2e$ and

$$\Phi_e^*(z) \mid \Phi_d^*(z) \quad \text{in } \mathbb{C}[z].$$

E. Bannai's lectures at OSU
in late 70s

P-poly. schemes = combinatorial analogue of compact 2-point homogeneous spaces

Q-poly. schemes = combinatorial analogue of compact symmetric rank 1 spaces

Hsien Chung Wang 1952

$$\left\{\begin{array}{l}\text{compact}\\ \text{2-point homogeneous}\\ \text{spaces}\end{array}\right\} = \left\{\begin{array}{l}\text{compact}\\ \text{symmetric spaces}\\ \text{of rank 1}\end{array}\right\}$$

Élie Cartan classified
compact symmetric spaces of rank 1

Bannai's Conjecture

(1) $\left\{\begin{array}{l}\text{primitive} \\ \text{P-poly. schemes} \\ \text{with sufficiently large} \\ \quad\text{diameter } d\end{array}\right\} = \left\{\begin{array}{l}\text{primitive} \\ \text{Q-poly. schemes} \\ \text{with sufficiently} \\ \text{large diameter } d\end{array}\right\}$

i.e. $P \Longleftrightarrow Q$ if primitive and $d >> 1$.

Def $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ is primitive if the graph (X, R_i) is connected for all i $(1 \le i \le d)$.

(2) If $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ is a (P and Q)-polynomial scheme with sufficiently large diameter d, then $\mathcal{X}$ is one of the following or their 'relative'.

(I)

(0) n-gon

(i) $J(v,k)$, $k \le \frac{v}{2}$ Johnson scheme

(ii) $J_q(v,k)$, $k \le \frac{v}{2}$ q-Johnson scheme

(iii) dual polar scheme

V: vector space over $\mathbb{F}_q$
with a non-degenerate form

X = the set of maximal totally isotropic subspaces

$$(x,y) \in R_i \iff \dim(x \cap y) = d-i$$

$B_d(q)$		$\dim V = 2d+1$	quadratic
$C_d(q)$		$2d$	symplectic
$D_d(q)$		$2d$	quadratic (Witt index d)
${}^2D_{d+1}(q)$		$2d+2$	quadratic (Witt index d)
${}^2A_{2d}(r)$	$(q=r^2)$	$2d+1$	Hermitian
${}^2A_{2d-1}(r)$	$(q=r^2)$	$2d$	Hermitian

(II)

(i) $H(d,q)$. Hamming scheme

(ii) $Bil_{d\times n}(q)$, $d \le n$, bilinear forms scheme

X = the set of $d \times n$ matrices over $\mathbb{F}_q$

$(x,y) \in R_i \iff \text{rank}\,(x-y) = i$, $0 \le i \le d$

(iii) affine schemes

(iii-1) $Alt_n(q)$, alternating bilinear forms scheme

V : n-dim vector space over $\mathbb{F}_q$

X = the set of alternating bilinear forms on V

$(x,y) \in R_i \iff \text{rank}\,(x-y) = 2i$, $0 \le i \le d = [\frac{n}{2}]$

(iii-2) $Her(r)$, Hermitean forms scheme

V : d-dim vector space over $\mathbb{F}_q$ $(q = r^2)$

X = the set of hermitean forms on V

$(x,y) \in R_i \iff \text{rank}\,(x-y) = i$

(iii-3) $Quad_n(q)$, quadratic forms scheme

V : n-dim vector space over $\mathbb{F}_q$

X = the set of quadratic forms on V

$(x,y) \in R_i \iff \text{rank}\,(x-y) = 2i-1, 2i$

$0 \le i \le d = [\frac{n+1}{2}]$

'relatives'

(a) bipartite half or antipodal quotient
of an imprimitive (P & Q)-poly. scheme

(b) 2nd P-polynomial structure
or 2nd Q-polynomial structure

(c) extended bipartite double
or fusion scheme of a (P&Q)-poly. scheme

(d) cospectral but not isomorphic scheme

Remark

(1) (iii-3) $Quad_n(q)$ was a conjecture
at the time of Bannai's lectures
and it was proved by Y. Egawa later.

(2) $Alt_n(q)$ and $Quad_{n-1}(q)$ are cospectral
but not isomorphic.

(3) $H(d,4)$ and Doob scheme are cospectral
but not isomorphic

newly found (P&Q)-poly schemes
after Bannai's conjecture

$Hem_d(q)$: Hemmeter scheme

$Ust_{[\frac{d}{2}]}(q)$: Ustimenko scheme

${}^2J_q(2d+1, d)$: twisted q-Johnson scheme
twisted Grassman scheme
van Dam - Koolen scheme
cospectral with $J_q(2d+1, d)$
but not isomorphic

$Hem_d(q)$, $D_d(q)$ cospectral

$Ust_{[\frac{d}{2}]}(q)$, $\frac{1}{2}D_d(q)$ cospectral

$$
\begin{tikzcd}
 & B_{d-1}(q) \arrow[dl, "\text{extended bipartite double}"'] \arrow[dr, "\text{dist. 1, 2 fusion}"] & \\
D_d(q) \arrow[rr, "\text{bipartite half}"] \arrow[dr, "\text{dist. 2}"'] & & \frac{1}{2}D_d(q) \arrow[dl, "{[\frac{d}{2}]\text{th subconstituent}}"] \\
 & Alt_d(q) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & C_{d-1}(q) \arrow[dl, "\text{extended bipartite double}"'] \arrow[dr, "\text{dist. 1, 2 fusion}"] & \\
Hem_d(q) \arrow[rr, "\text{bipartite half}"] \arrow[dr, "\text{dist. 2}"'] & & Ust_{[\frac{d}{2}]}(q) \arrow[dl, "{[\frac{d}{2}]\text{th subconst.}}"] \\
 & Quad_{d-1}(q) &
\end{tikzcd}
$$

$\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$
(P & Q)-poly. scheme

$A_i = v_i(A_1)$

$nE_i = v_i^*(nE_1)$ w.r.t. Hadamard product $\circ$
$(n = |X|)$

Let

$$A_1 = \theta_0 E_0 + \theta_1 E_1 + \cdots + \theta_d E_d$$

$$nE_1 = \theta_0^* A_0 + \theta_1^* A_1 + \cdots + \theta_d^* A_d$$

and set

$$k_i = v_i(\theta_0), \quad 0 \le i \le d$$

$$m_i = v_i^*(\theta_0^*), \quad 0 \le i \le d .$$

orthogonal polynomials $\{v_i(x)\}_{i=0}^{d}$

$$\sum_{\nu=0}^{d} v_i(\theta_\nu) v_j(\theta_\nu) m_\nu = \delta_{ij} n k_i$$

orthogonal polynomials $\{v_i^*(x)\}_{i=0}^{d}$

$$\sum_{\nu=0}^{d} v_i^*(\theta_\nu^*) v_j^*(\theta_\nu^*) k_\nu = \delta_{ij} n m_i$$

dual

$$\frac{v_i(\theta_j)}{k_i} = \frac{v_j^*(\theta_i^*)}{m_j}$$

Leonard Theorem

If the orthogonal polynomials $\{v_i(x)\}_{i=0}^{d}$ and $\{v_i^*(x)\}_{i=0}^{d}$ are dual in the above sense, then they are Askey-Wilson polynomials (q-Racah polynomials) or their limits.

No. P & Q - poly.
Date

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For details, see

E. Bannai - T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings, Menlo Park, California (1984)

Theorem For a (P and Q)-polynomial scheme with sufficiently large diameter d,

$\theta_0, \theta_1, \cdots, \theta_d \in \mathbb{Z}$.

(G. Dickie) enough to assume $d \geq 5$.

Terwilliger algebra

P. Terwilliger, The subconstituent algebra of an association scheme I, II, III, J. Alg. Comb. 1 (1992), 2 (1993),

$\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ (P and Q)-poly. scheme

$\mathfrak{A} = \langle A_0, A_1, \cdots, A_d \rangle$

$= \langle E_0, E_1, \cdots, E_d \rangle$ primitive idempotents

Bose-Mesner algebra

$V = \mathbb{C}^X = \{ f : X \longrightarrow \mathbb{C} \}$

standard module
for the full matrix alg. $M_X(\mathbb{C})$

$V = V_0 + V_1 + \cdots + V_d$ direct sum

$V_i = E_i V$

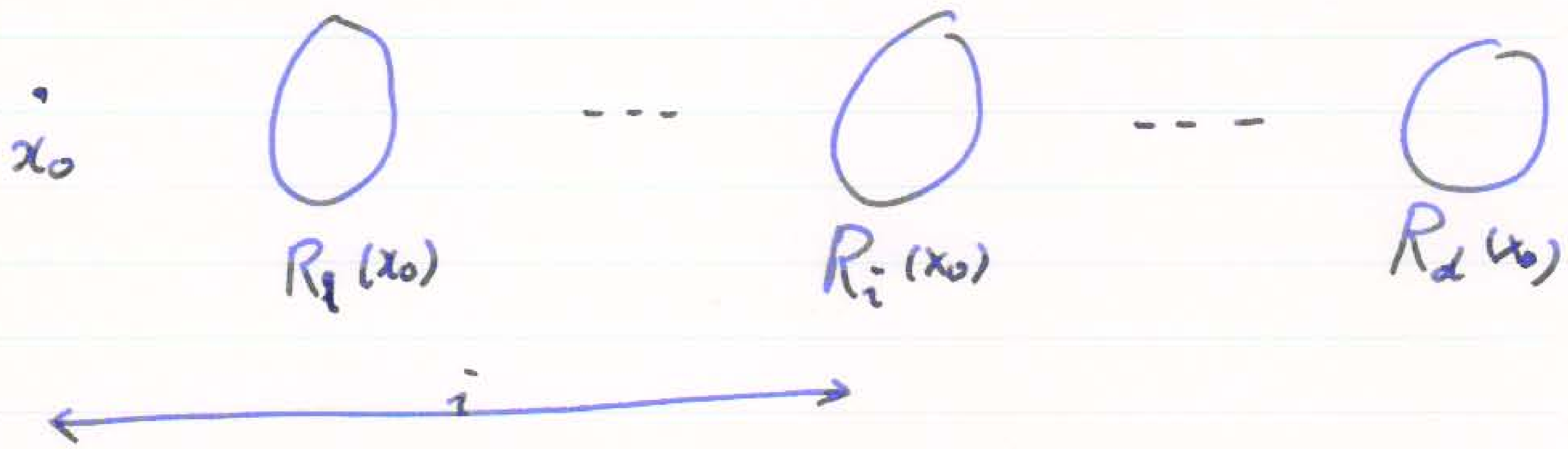
$A = A_1$ has eigenvalue θ_i on V_i

$A = \theta_0 E_0 + \theta_1 E_1 + \cdots + \theta_d E_d$

Fix $x_0 \in X$ (base point)

and set

$$V_i^* = V_i^*(x_0) = \left\{ f : X \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} f(x) = 0 \\ \text{if } (x_0, x) \notin R_i \end{array} \right\}$$



$V_i^* \ni f \iff f$ vanishes outside $R_i(x_0)$

$$V = \bigoplus_{i=0}^{d} V_i^*$$

$E_i^* = E_i^*(x_0) : V \longrightarrow V_i^*$ projection.

$$T = T(x_0) = \langle E_i, E_j^* \mid 0 \le i, j \le d \rangle$$

the subalgebra of $M_X(\mathbb{C})$
generated by E_i, E_j^*, $0 \le i, j \le d$

called the <u>subconstituent alg.</u>

<u>Terwilliger alg.</u>

$$A_1 = \theta_0 E_0 + \theta_1 E_1 + \cdots + \theta_d E_d, \quad E_i E_j = \delta_{ij} E_i$$

$$n E_1 = \theta_0^* A_0 + \theta_1^* A_1 + \cdots + \theta_d^* A_d, \quad A_i \circ A_j = \delta_{ij} A_i$$

Set

$$A_1^* = \theta_0^* E_0^* + \theta_1^* E_1^* + \cdots + \theta_d^* E_d^*, \quad E_i^* E_j^* = \delta_{ij} E_i^*$$

Then

$$A_i = v_i(A_1) \quad \text{in } M_X(\mathbb{C})$$

$$n E_i = v_i^*(n E_1) \quad \text{w.r.t. the Hadamard product } \circ$$

$$A_i^* = v_i^*(A_1^*) \quad \text{in } M_X(\mathbb{C})$$

$$A_i^* A_j^* = \sum_{k=0}^{d} q_{ij}^k A_k^*$$

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$$T = T(x_0) = \langle A, A^* \rangle \subset M_X(\mathbb{C})$$

generated by $A = A_1$, $A^* = A_1^*$.

non commutative
semi-simple algebra

$V = \mathbb{C}^X \supset W$ irreducible T-module

We get a $\underline{\text{TD-pair}}$ (tridiagonal pair)
system above TD-pair; system above pair

$A|_W$, $A^*|_W \in \mathrm{End}(W)$: diagonalizable

$$W = \bigoplus_{i=0}^{r} W_i, \quad \text{e.s. decomp. of } A$$

$$W = \bigoplus_{i=0}^{r} W_i^*, \quad \text{e.s. decomp. of } A^*$$

(i) $A W_i^* \subseteq W_{i-1}^* + W_i^* + W_{i+1}^*$, $0 \le i \le r$

$W_{-1}^* = 0$, $W_{r+1}^* = 0$

(ii) $A^* W_i \subseteq W_{i-1} + W_i + W_{i+1}$, $0 \le i \le r$

$W_{-1} = 0$, $W_{r+1} = 0$

(iii) W is irreducible as an $\langle A, A^* \rangle$-module.

TD-pair is an <u>L-pair</u> (Leonard pair)

if $\dim W_i = \dim W_i^* = 1$, $0 \le i \le r$.

<u>Leonard Theorem</u>: Classification of orthogonal polynomials that are dual each other

$\updownarrow$ equivalent

<u>Terwilliger Theorem</u>: Classification of L-pairs (systems)

Ito - Terwilliger: Classification of TD-pairs (systems)

TD-pair = 'tensor product' of L-pairs

character formula

$$ch(\lambda) \underset{def}{=} \sum_{i=0}^{r} (\dim W_i) \lambda^i$$

Then

$$ch(\lambda) = \prod_{i=1}^{n} (1+\lambda+\lambda^2+\cdots+\lambda^{l_i})$$

for some $n \in \mathbb{N}$, $l_1, l_2, \cdots, l_n \in \mathbb{N}$.

case $n=1$, $l_1 = r$: L-pair

$\dim W_i = 1$, $0 \leq i \leq r$

case $n=2$, $l_1 = l$, $l_2 = 2$

$$\begin{aligned} ch(\lambda) &= (1+\lambda+\cdots+\lambda^l)(1+\lambda) \\ &= 1+2\lambda+\cdots+2\lambda^l+\lambda^{l+1} \end{aligned}$$

$\dim W_i = 2$, $1 \leq i \leq l$

$\dim W_0 = 1$, $\dim W_{l+1} = 1$ $(r = l+1)$

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TD-relation (tridiagonal relations)

$$A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3$$
$$= \gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A)$$

for some $\beta, \gamma, \delta \in \mathbb{C}$

$$A^{*3}A - (\beta+1)(A^{*2}AA^* - A^*AA^{*2}) - AA^{*3}$$
$$= \gamma^*(A^{*2}A - AA^{*2}) + \delta^*(A^*A - AA^*)$$

for some β, $\gamma^*, \delta^* \in \mathbb{C}$

$(\beta^* = \beta)$

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$\beta = q^2 + q^{-2}$

type I : $\beta \neq \pm 2$ $(q^2 \neq \pm 1)$

type II : $\beta = 2$ $(q^2 = 1)$

type III : $\beta = -2$ $(q^2 = -1)$

generic case : type I $q \neq$ root of unity

The irreducible T-module W
is obtained by a finite-dimensional
irreducible representation of
the quantum affine algebra $U_q(\hat{sl}_2)$

$$A|_W ,\ A^*|_W \xhookrightarrow[\text{embedded}]{} U_q(L(sl_2))|_W$$

$$U_q(L(sl_2)) = U_q(\hat{sl}_2) / (k_0 k_1 - 1)$$

Classification of (P and Q)-poly. schemes

(A) Determine $\{v_i(x)\}_{i=0}^{d}$, $\{v_i^*(x)\}_{i=0}^{d}$:

not all of the Askey-Wilson polynomials (or their limits) appear

In other words, determine the parameters that can appear for (P and Q)-poly. schemes

(and show they are in Bannai's list)

(B) Characterize (P and Q)-poly. schemes by the parameters.

No. classification
Date

Done

(B) $H(d,q)$, Doob scheme : Egawa

$J(v,d)$: Terwilliger, Neumaier

${}^2A_{2d-1}(r)$: Ivanov - Shpectrov

$Her_d(r)$: Ivanov - Shpectrov (Terwilliger)

$J_q(v,d)$, $3 \le d \le \frac{v}{2}$, Metsch
except for $q \ge 4$, $v = 2d, 2d+1$
$q = 3$, $v = 2d, 2d+1, 2d+2$
$q = 2$, $v = 2d, 2d+1, 2d+2, 2d+3$

$Bil_{d \times n}(q)$, $d \le n$, Metsch
except for $q \ge 3$, $n = d, d+1, d+2$
$q = 2$, $n = d, d+1, d+2, d+3$

(A) type II, type III : Terwilliger

Open (A) type I
(B) dual polar scheme,
affine scheme

$J_q(v,d)$

$Bil_{d \times n}(q)$
that are not covered above.

(A) : $V = \mathbb{C}^X \supset W$ irreducible T-module

$e = \mathrm{Min}\{ i \mid E_i^* W \neq 0 \}$
endpoint

$e=0$. W : principal T-module
uniquely determined

$e=1$ A_W, A_W^* :
either L-pair
or a tensor product of
2 L-pairs, one of
which has diameter 1

$e=2$ A_W, A_W^* :
either L-pair
or ----- (strong restriction
on the tensor product of
L-pairs)

$\rightsquigarrow$ local structure on
1st subconstituent $R_1(x_0)$
2nd " $R_2(x_0)$

(B) parameter

$+$

local structure on $R_1(x_0) + R_2(x_0)$

$\rightsquigarrow$ global structure